Parameter identification technique for uncertain chaotic systems using state feedback and steady-state analysis

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A technique is introduced for identifying uncertain and/or unknown parameters of chaotic dynamical systems via using simple state feedback. The proposed technique is based on bringing the system into a stable steady state and then solving for the unknown parameters using a simple algebraic method that requires access to the complete or partial states of the system depending on the dynamical model of the chaotic system. The choice of the state feedback is optimized in terms of practicality and causality via employing a single feedback signal and tuning the feedback gain to ensure both stability and identifiability. The case when only a single scalar time series of one of the states is available is also considered and it is demonstrated that a synchronization-based state observer can be augmented to the state feedback to address this problem. A detailed case study using the Lorenz system is used to exemplify the suggested technique. In addition, both the Rössler and Chua systems are examined as possible candidates for utilizing the proposed methodology when partial identification of the unknown parameters is considered. Finally, the dependence of the proposed technique on the structure of the chaotic dynamical model and the operating conditions is discussed and its advantages and limitations are highlighted via comparing it with other methods reported in the literature.

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I. INTRODUCTION

The topic of modeling and time series analysis of chaotic physical processes with known structures has been an active area of research for the last two decades [1–15], where various techniques have been reported in the literature. Usually it is required to estimate some or all of the chaotic system parameters and consequently the degree of complexity of the estimation process depends crucially on many factors; among them the structure and type of nonlinearity of the system at hand, complete or partial availability of the states for direct measurement, and the nature of the application, e.g., whether it is required to control the chaos, or to synchronize two identical or different chaotic systems. Using a synchronization-based approach and minimizing the averaged synchronization error was used in [1] to build a parameter estimator that recovers the parameter values of a given model from a single time series. This technique can be successfully applied to very large scale integrated (VLSI) implementations of electronic systems such as Chua's circuit where the determination of the actual values of the individual components is very difficult. In [2], another autosynchronization approach was used to come up with a practical method for deriving the necessary ordinary differential equations for the parameter controlling loop. Although it was demonstrated that this technique offers various potential real-time applications, it is argued that the choice of the parameter update laws and proving their stability and convergence is not an easy task, and remains a bottleneck when generalizing its application to different classes of chaotic systems. A simple method to reveal the parameters of the Lorenz system was introduced in [3] using maps, as measures of the errors between the states in the drive and response models of the chaotic system, for updating and estimating the unknown parameters. Dynamic minimization using a combination of two different control methods, linear feedback for synchronizing system variables and adaptive control, was explored in [4]. Although this technique was demonstrated to be reasonably stable against noise, it was shown that it will work only for a given time series and fail for others. A similar result was reported in [5], where it was necessary to have access to the complete state vector of the chaotic system in order to estimate the unknown parameters. Different applications in plasma physics and secure communications were covered in [4,5] to demonstrate the effectiveness of the proposed techniques. Another method, based on random optimization, was introduced in [6] where the parameters were randomly searched for in a sequential manner as the degree of the chaos synchronization is increased. This method was applied to the Lang-Kobayashi model for the chaotic semiconductor laser. Variational calculus was also considered in [7] to develop an analytical framework for the robust design of dynamical systems that guarantees online estimation of all model parameters of a given chaotic-hyperchaotic system. This method was shown to be appropriate for a more realistic situation where only discrete time measurements of the experimental output are available. Using the invariance principle of differential equations, all unknown model parameters were estimated dynamically in [8], where access to the full state vector of the drive system was required. This method employed adjustable feedback strength between the drive and response systems using a systematic synchronization-based approach. A method to partially identify the parameters of the Lorenz system was introduced in [9] where the model was reformulated in such a way to facilitate the application of an adaptive parameter identifier, and the estimated parameter was directly used in a state estimator for the purpose of synchronization. A numerical approach with fast convergence was introduced in [10] where the adjustable parameter is the only control input of the secondary system that needs to be synchronized with the original system. Although this method produced approximated results, it has the advantage of being systematic and does not require explicit coupling. A combination of slide mode control and linear feedback control was used in [11] using master-slave synchronization where the master system is transformed into a standard form with zero dynamics via employing geometric control. In [12] it was pointed out that some of currently reported results on the synchronization-based parameter identification of dynamical systems from time series are incomplete and a linear independence condition was introduced, which is sufficient for such parameter identification of general dynamical systems. Other results that deal with the problem of complete and partial identification can be found in [13–15].

It is interesting to notice that most, if not all, parameter identification algorithms come together with some form of synchronization, i.e., it is required to identify the unknown parameter(s) in order to achieve synchronization, or synchronization is used as an intermediate step to identify the unknown parameter(s). Ever since the early work in [16,17] chaos synchronization has received great attention due to its potential applications [18]. Another important application for parameters identification is chaos control [19]. The design procedure for either synchronization or control usually achieves the desired objectives by constructing a suitable Lyapunov function and forcing its derivative to be negative definite. However, the construction of Lyapunov functions remains to be a difficult task, and is usually considered a bottleneck in the design of the control law [20–22]. Achieving the stability and convergence of the parameter identification algorithms is usually difficult to prove analytically [23], especially when using synchronization due to the increased order of the overall system [24,25].

In this paper a different approach is adopted where the difficult design of dynamic parameter update laws is replaced by a simple algebraic method. In addition, synchronization can be avoided provided that access to the complete state vector of the chaotic system is possible. The main contribution of this work is to provide a simple method for parameter identification that results in a low order model easily implementable using both analog and digital hardware with possible applications to secure communication. This work is motivated by the results reported in [26] where failure of using adaptive synchronization methods for the purpose of parameter identification was addressed and [27] where the ability of perturbation signals to redirect the system's trajectory flow within its phase space was examined.

The rest of this paper is organized as follows: Sec. II introduces the mathematical model of the Lorenz system and investigates the problem of complete identification of the parameters assuming availability of all the states. In addition, Sec. II studies different equilibrium conditions that correspond to different implementations of the stabilizing state feedback controller. Tuning of the feedback gain is addressed in Sec. III and a detailed analysis of the effect of the feedback gain on both stability and identifiability is carried out for the Lorenz system for a particular case of one of the possible implementations of the proposed stabilizing feedback controller. Section IV investigates partial identification of some of the unknown parameters of the Lorenz system when only a single scalar time series for one of the states is available. A special case study is introduced in Sec. IV A, while the effect of partial identification on complete synchronization of identical Lorenz systems is investigated in Sec. IV B. Extending the proposed methodology of partial identification to other chaotic systems is explored in Sec. V, where both the Rössler and Chua systems are studied in Secs. V A and V B, respectively. Comments regarding the practicality, advantages, and limitations of the proposed technique for both complete and partial identification are highlighted in Sec. VI as well as a brief comparison with other techniques reported in the literature. Finally, a conclusion is presented that summarizes the work done in this paper while stating possible extensions for it.

II. COMPLETE IDENTIFICATION OF THE LORENZ SYSTEM USING FULL STATE FEEDBACK

The Lorenz system is considered to be a benchmark model when referring to chaos and its synchronization-based applications [2–12]. Although the Lorenz "strange attractor" was originally noticed in weather patterns [28], other practical applications exhibit such strange behavior, e.g., single-mode lasers [29], thermal convection [30], and permanent magnet synchronous machines [31]. In this section, the Lorenz model is used to exemplify the suggested technique of both complete and partial identification of the unknown or uncertain parameters of a chaotic system. The mathematical model of the Lorenz system is assumed to take the form

$$\dot{x}_1 = -\sigma x_1 + \sigma x_2 + u,$$

$$\dot{x}_2 = \rho x_1 - x_2 - x_1 x_3,$$

$$\dot{x}_3 = -\beta x_3 + x_1 x_2, \tag{1}$$

where the nominal values of the parameters are 10.0, 20.0, and 1.0 corresponding to σ, ρ, and β, respectively, and $u = kx_i$, $i \in \{1,2,3\}$. These values are known to produce chaos for the free running case, i.e., $k=0$. Using linear analysis, the Jacobian matrix J of the controlled system at equilibrium is given by

$$J = \begin{bmatrix} -\sigma + \dfrac{\partial u}{\partial x_1} & \sigma + \dfrac{\partial u}{\partial x_2} & \dfrac{\partial u}{\partial x_3} \\ \rho - x_3 & -1 & -x_1 \\ x_2 & x_1 & -\beta \end{bmatrix}_{eq}, \tag{2}$$

where the subscript "eq" stands for equilibrium. When a single state feedback is used, the system can settle down to a steady state that depends on the structure of the control signal, u. Using the dynamical model of the Lorenz system, in Eq. (1), and excluding the origin, the equilibrium points are given by

$$X_{eq}=\begin{bmatrix}\frac{\pm 1}{\sigma-k}\sqrt{(\sigma-k)\beta[\sigma(\rho-1)+k]}\\ \frac{\pm 1}{\sigma}\sqrt{(\sigma-k)\beta[\sigma(\rho-1)+k]}\\ (\rho-1)+\frac{k}{\sigma}\end{bmatrix} \tag{3}$$

for $u=kx_1$,

$$X_{eq}=\begin{bmatrix}\pm\sqrt{\beta(\rho-1)+\frac{\rho\beta k}{\sigma}}\\ \pm\frac{\sigma}{\sigma+k}\sqrt{\beta(\rho-1)+\frac{\rho\beta k}{\sigma}}\\ \left(\rho-\frac{\sigma}{\sigma+k}\right)\end{bmatrix} \tag{4}$$

for $u=kx_2$, and

$$X_{eq}=\begin{bmatrix}\pm\sqrt{\beta(\rho-1)+\left(\frac{\rho k}{2\sigma}\right)^2}+\left(\frac{\rho k}{2\sigma}\right)\\ \left(\frac{\beta\sigma^2}{\beta\sigma^2+k^2}\right)\left[\pm\sqrt{\beta(\rho-1)+\left(\frac{\rho k}{2\sigma}\right)^2}-\left(\frac{(\rho-2)k}{2\sigma}\right)\right]\\ \left(\frac{\beta\sigma^2}{\beta\sigma^2+k^2}\right)\left[(\rho-1)+\frac{k}{\sigma\beta}\left\{\pm\sqrt{\beta(\rho-1)+\left(\frac{\rho k}{2\sigma}\right)^2}-\left(\frac{\rho k}{2\sigma}\right)\right\}\right]\end{bmatrix} \tag{5}$$

for $u=kx_3$. It can be verified that when $k=0$ all equilibrium points, given in Eqs. (3)–(5), reduce to the $[\pm\sqrt{\beta(\rho-1)}\ \ \pm\sqrt{\beta(\rho-1)}\ \ (\rho-1)]^T$, which is typical for the free running chaotic Lorenz system. For notation convenience, the zero steady state is denoted by X_{eq0}, and the other two equilibrium points are denoted by X_{eq1} and X_{eq2}, corresponding to the positive and negative signs of the square root in Eqs. (3)–(5), respectively. Solving Eq. (1) for steady state reveals the interesting result depicted by Eq. (6), which is a special characteristic of the Lorenz equation, that the steady-state solution has a unique form regardless of which state is used to constitute the feedback,

$$\hat{\sigma}=\frac{u}{x_1-x_2},$$

$$\hat{\rho}=x_3+\frac{x_2}{x_1},$$

$$\hat{\beta}=\frac{x_1x_2}{x_3}. \tag{6}$$

Careful examination of Eq. (6) illustrates the need to avoid X_{eq0}, to ensure identifiability of the system. In addition the only feedback gain must be carefully chosen to have negative real parts for the Lyapunov exponents of the closed loop system. Figures 1–3 show the simulation results of the evolution of the identified parameters of the Lorenz system using $u=kx_1$, $u=kx_2$, $u=kx_3$, respectively. The state feedback controller was switched on after 50 s and the feedback gain was adjusted to −10, −5, and +10, respectively. These results illustrate the effectiveness of the proposed methodology as the estimates converged to their true values in a reasonable time that is of the same order as the dominant period of the original chaotic system.

To demonstrate the simplicity of the proposed technique, it is now applied to the Lorenz system with different chaotic set of parameters ($\alpha=10$, $\rho=28$, $\beta=8/3$) using the analog implementation shown in Fig. 4(a). For practical reasons and to meet the linearity constraints of the analog components used in constructing the circuit, a linear transformation was applied to Eq. (1) such that $u=0.2x_1$, $v=0.2x_2$, and $w=0.1x_3$ are the new states of the system. In addition, a time scaling was applied such $t\leftarrow t/\tau$ where $\tau=100$ μs. The control signal $u=kx_3$ ($k=10$) was applied to the chaotic system after 50 ms to identify β (which is scaled up 2.5 times) to conform to both Eq. (6) and the linear transformation used to transfer the system states (x_1, x_2, and x_3) to (u, v, and w), respectively. As illustrated in Figs. 4(b)–4(d) the chaotic performance was reduced to a stable steady-state condition and the response for $\hat{\beta}$ was very close to the results outlined in Fig. 3(b) where the steady state-value was 99.5% accurate.

III. TUNING OF THE FEEDBACK GAIN, *k*

The feedback gain must be tuned in order to achieve both stability and identifiability. Using linear system analysis, the Jacobian matrix J, given in Eq. (2), can be evaluated at the corresponding equilibrium point(s) to check for stability. In addition, the state feedback will cause the system to settle down to a steady-state equilibrium point that depends on initial conditions, structure of the feedback, and the feedback gain. Successful identification requires stabilizing the system to either X_{eq1} or X_{eq2}. Figure 5 shows the analysis of both stability and identifiability of the Lorenz system using different structures of the state feedback where the regions for instability are highlighted, while plotting the maximum

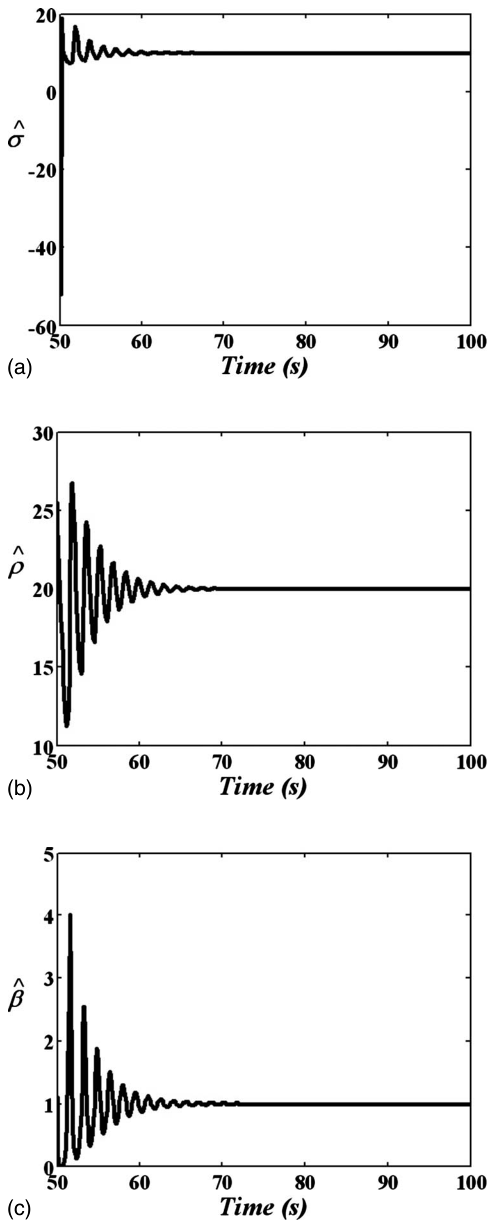

FIG. 1. Complete identification using $u=kx_1$.

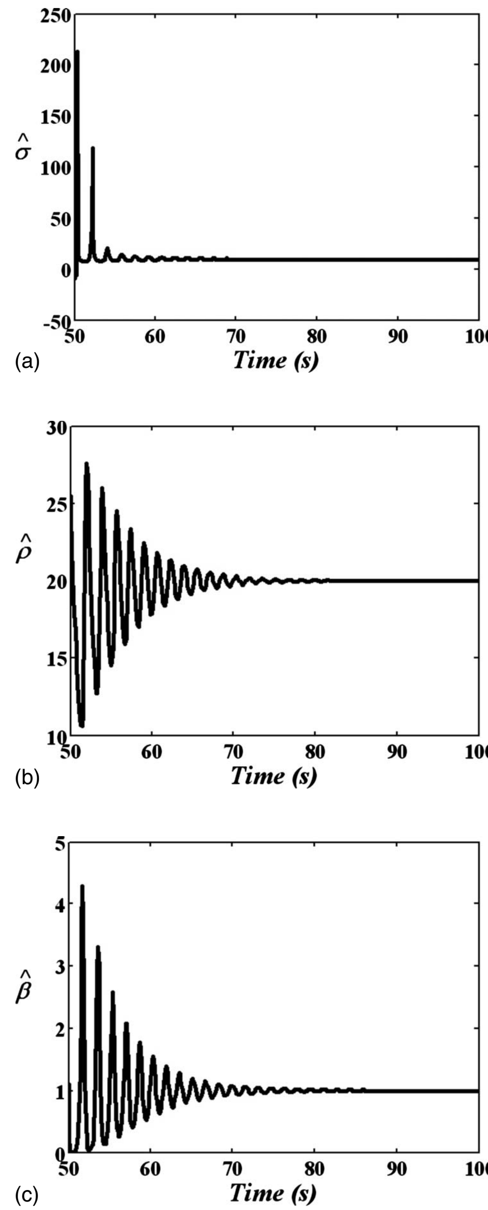

FIG. 2. Complete identification using $u=kx_2$.

Lyapunov exponent, λ_{max}, as a function of the feedback gain for stable regions. It is interesting to notice that all the curves for λ_{max} intersect with the boundary of the instability region at -12.1132 regardless of the value of k, which is identical to the free running chaotic system, i.e., $k=0$. In Fig. 5(a), the feedback signal was considered to be $u=kx_1$ and it was determined that the condition for stability reduces to $k<0$ and that the equilibrium point $[0\ 0\ 0]^T$ was never stable; hence identifiability is guaranteed as the system will always settle down to either X_{eq1} or X_{eq2} depending on initial conditions. It was also found that the λ_{max} asymptotically approaches k when the feedback gain is increased. Figure 5(b) illustrates the case when $u=kx_2$, where it was found that the system is always unstable for $k>-1.25$. However as k falls below -9.5 the only stable equilibrium point becomes the origin, regardless of initial conditions, which means the loss of identifiability. Thus the condition to achieve both stability and identifiability reduces to $-9.5<k<-1.25$. The final case when $u=kx_3$ was shown to have a very small region for instability corresponds to the condition $|k|<1$, as shown in Fig. 5(c), and that λ_{max} asymptotically approaches -10.0 as $|k|$ is increased. The origin was never stable under this structure of the state feedback and the system settled down to any of the remaining identifiable equilibrium points, X_{eq1} or X_{eq2}, depending on initial conditions.

When implementing this technique in either analog or digital hardware, the maximum permissible value of k should be constrained to avoid overdriving the system and introducing additional nonlinearity that might cause the overall system to be unstable. The value of k that results in the best compromise between agility and stability can be chosen if experimental results relating k to the settling time of the parameters estimator exist along with the maximum control

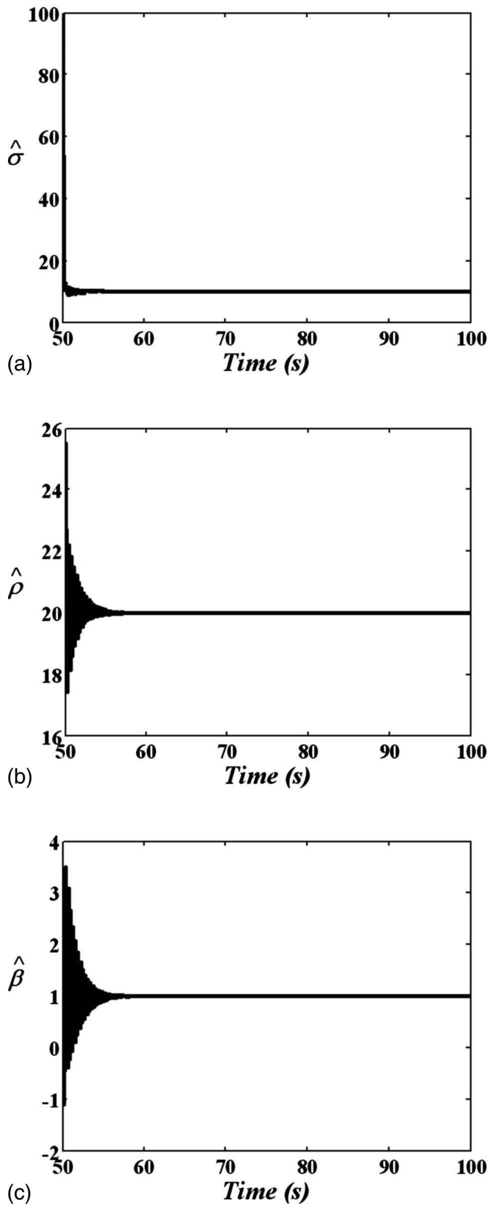

FIG. 3. Complete identification using $u=kx_3$.

effort (e.g., saturation voltage of operational amplifiers). Figure 6 shows the settling time of the parameters estimator (2% criteria) in the case when $u=kx_1$ illustrating that the convergence rate of the estimated parameters is monotonically decreasing when $|k|$ is increased. A similar result can be obtained for the other two cases when using either x_2 or x_3 for feedback.

IV. PARTIAL IDENTIFICATION OF THE PARAMETERS

The steady-state analysis carried out in the previous section considered only the cases where a single feedback was used in the dynamic equation of x_1. Each of these cases used a simple linear feedback; however, other alternatives exist, e.g., using two or more feedback signals or using nonlinear feedback. Each case should be investigated individually to find the permissible range of the feedback gain(s) that brings the original chaotic system to a stable and identifiable condition. In practice some constraints might be imposed on which time series to use either for feedback, measurement, or both. This case is now investigated by trying to partially identify one of the system parameters using a single scalar time series.

A. Identifying σ giving knowledge of ρ, β, and x_1 only

Assuming that both ρ and β are known and that only x_1 is available for feedback, and with reference to Eq. (6), it will be required to design a state observer for x_2 to be able to identify σ. This can be done via using a synchronization-based state observer as illustrated in Eq. (7),

$$\dot{x}_1 = -\sigma x_1 + \sigma x_2 + u,$$

$$\dot{x}_2 = \rho x_1 - x_2 - x_1 x_3,$$

$$\dot{x}_3 = -\beta x_3 + x_1 x_2,$$

$$\dot{\hat{x}}_2 = \rho x_1 - \hat{x}_2 - x_1 \hat{x}_3,$$

$$\dot{\hat{x}}_3 = -\beta \hat{x}_3 + x_1 \hat{x}_2, \tag{7}$$

where $\hat{x}_2$ and $\hat{x}_3$ are the estimates of x_2 and x_3, respectively. The stability and convergence of the state observer are proved using the Lyapunov function L, given in Eq. (8),

$$L = (\hat{x}_2 - x_2)^2 + (\hat{x}_3 - x_3)^2, \tag{8}$$

for which we have the result illustrated in Eq. (9) verifying negative definiteness of $\dot{L}$,

$$\begin{aligned}\dot{L} &= 2(\hat{x}_2 - x_2)(\dot{\hat{x}}_2 - \dot{x}_2) + 2(\hat{x}_3 - x_3)(\dot{\hat{x}}_3 - \dot{x}_3)\\ &= -2(\hat{x}_2 - x_2)^2 - 2\beta(\hat{x}_3 - x_3)^2 < 0.\end{aligned} \tag{9}$$

The estimate of the unknown parameter σ is now given by

$$\hat{\sigma} = \frac{u}{x_1 - \hat{x}_2}, \tag{10}$$

where $u \in \{kx_1, k\hat{x}_2, k\hat{x}_3\}$. Figure 7 demonstrates the results of the system, when $u=-10x_1$, where the synchronization errors ($\hat{e}_i=\hat{x}_i-x_i, i=1,2$) are shown to vanish rapidly in Fig. 7(a). Figure 7(b) further illustrates the effectiveness of augmenting the synchronization-based state observer to the state feedback controller as it is clear that $\hat{\sigma}$ converges rapidly to its true value.

B. Using partial identification for complete synchronization

The results obtained in the previous section can be extended to accommodate other scenarios for which the uncertain parameter appears in the dynamic equation of the only available time series of the system, i.e., identifying σ given x_1, ρ given x_2, or β given x_3. In each of these cases the knowledge of the remaining two parameters can be used to design a state observer for the immeasurable two states of the

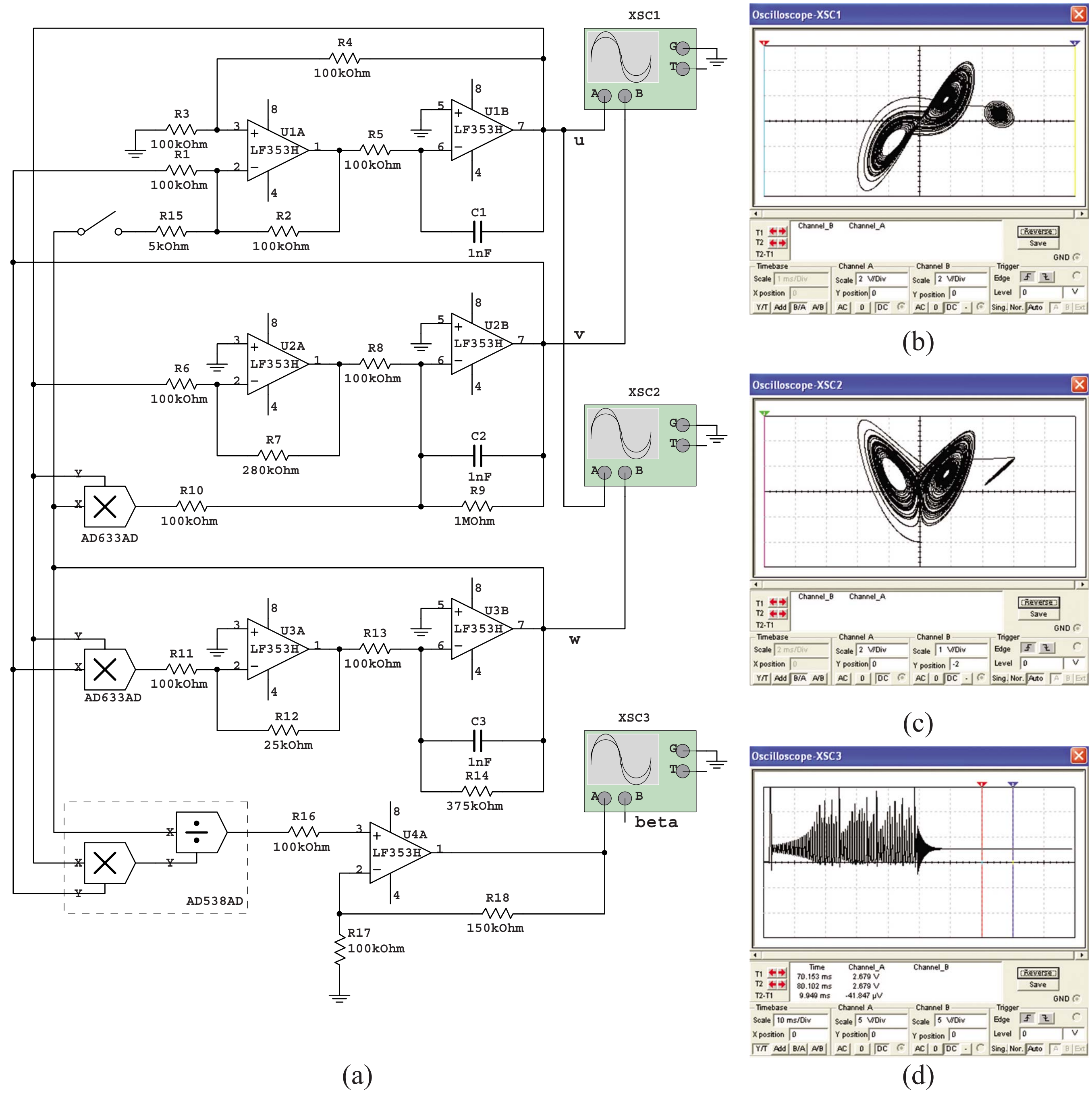

FIG. 4. (Color online) Analog implementation of the proposed technique. The analog circuit is shown in (a), while the phase planes for v (vertical axis: 2 V/division) against u (horizontal axis: 2 V/division) and w (vertical axis: 2 V/division) against u (horizontal axis: 1 V/division) are shown in (b) and (c), respectively. The estimation of β is shown in (d) where the value (vertical axis: 5 V/division) is drawn against time (horizontal axis: 10 ms/division). The chaotic performance was maintained for the first 50 ms, and then the switch in (a), connected to R_{15}, was closed to initiate the estimation process, via employing the state feedback, and as illustrated in (b)–(d), a stable steady state was obtained.

system while using the state feedback to stabilize the system to an identifiable equilibrium point. When trying to completely synchronize two identical Lorenz systems with mismatched parameters using drive-response mechanism, a synchronization offset can persist in the response system [32]. As an extension for the previously considered case where only σ was assumed uncertain and using a full state synchronization-based observer, the dynamics of the response system are given by

$$\dot{\hat{x}}_1 = -\sigma_n \hat{x}_1 + \sigma_n \hat{x}_2 + k x_1,$$

$$\dot{\hat{x}}_2 = \rho x_1 - \hat{x}_2 - x_1 \hat{x}_3,$$

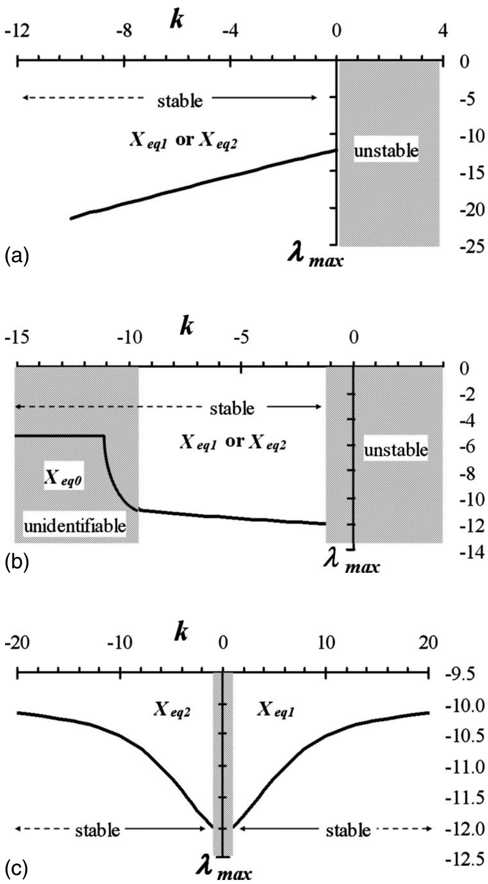

FIG. 5. Stability and identifiability regions for the state feedback controller.

$$\dot{\hat{x}}_3 = -\beta \hat{x}_3 + x_1 \hat{x}_2, \tag{11}$$

where σ_n is the nominal value of the uncertain parameter σ. It is very difficult to analytically prove the stability of this system using a Lyapunov-based approach; however, careful examination of the individual dynamics of the Eq. (11) shows that the both $\hat{x}_2$ and $\hat{x}_3$ are unaffected by $\hat{x}_1$. Thus the previous results from Eqs. (7)–(10) are still applicable, and consequently

$$\hat{x}_2 \to x_2, \quad \hat{x}_3 \to x_3,$$

$$e_{1ss} = (\hat{x}_1 - x_1)_{ss} = k\left(\frac{1}{\sigma_n} - \frac{1}{\sigma}\right) x_{1ss},$$

$$\hat{\sigma} = \frac{ku}{x_1 - \hat{x}_2}, \tag{12}$$

where the subscript "*ss*" stands for steady state. It is interesting to note that the synchronization error for this particular case can be made zero by setting the feedback to zero, i.e., using the free-running case of the original chaotic Lorenz system. This agrees with the experimental results reported in [32]. Figure 8 illustrates these results as it is shown that both $\hat{x}_2$ and $\hat{x}_3$ converged to their true values while having an offset in $\hat{x}_1$, for the case $u = -10x_1$, as depicted by Eq. (12).

Another case when both σ and β are known while only the scalar time series for x_2 is available is now considered. The dynamics of the response system is given by

$$\dot{\hat{x}}_1 = -\sigma \hat{x}_1 + \sigma x_2 + k x_2,$$

$$\dot{\hat{x}}_2 = \rho_n \hat{x}_1 - \hat{x}_2 - \hat{x}_1 \hat{x}_3,$$

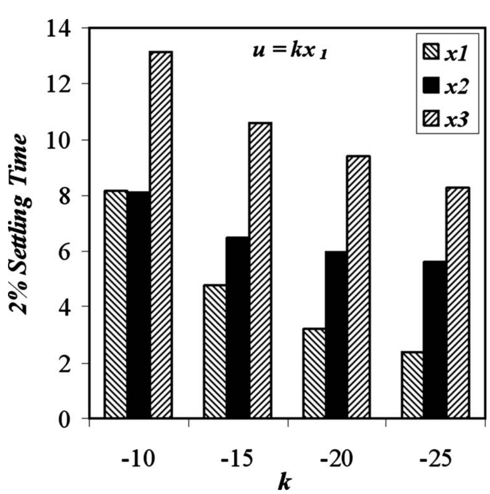

FIG. 6. Settling time (2%) as a function of k.

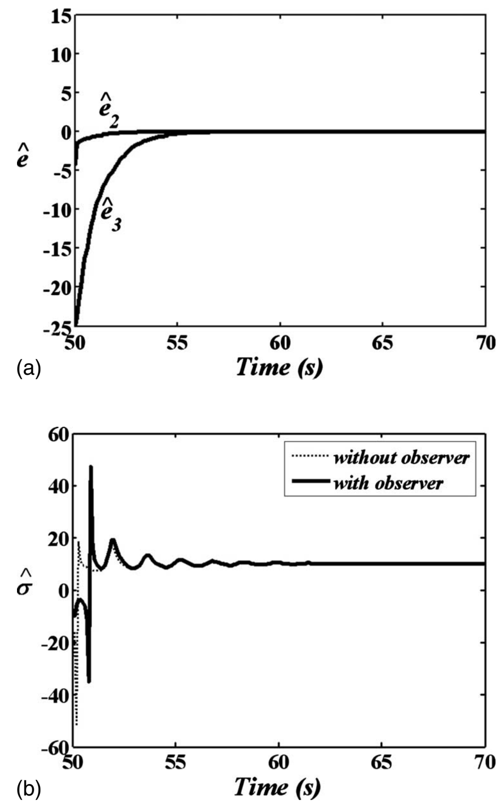

FIG. 7. Synchronization errors for x_2 and x_3 and partial identification of σ.

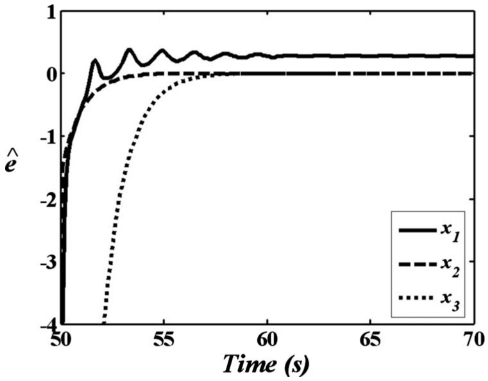

FIG. 8. Case study I: using x_1 for synchronization.

$$\dot{\hat{x}}_3 = -\beta\hat{x}_3 + \hat{x}_1 x_2, \tag{13}$$

where ρ_n is the nominal value of the uncertain parameter ρ. Once again, both $\hat{x}_1$ and $\hat{x}_3$ are unaffected by $\hat{x}_2$ and it is straightforward to prove that $(\dot{\hat{x}}_1 - \dot{x}_1) = -\sigma(\hat{x}_1 - x_1)$ and $(\dot{\hat{x}}_3 - \dot{x}_3) = -\beta(\hat{x}_3 - x_3) + x_2(\hat{x}_1 - x_1)$. This means that the synchronization error for x_1 will exponentially decay to zero and consequently the synchronization error for x_3 will follow. The complete asymptotic behavior of the response system is given by

$$\hat{x}_1 \to x_1, \quad \hat{x}_3 \to x_3,$$

$$e_{2ss} = (\hat{x}_2 - x_2)_{ss} = (\rho_n - \rho) x_{1ss},$$

$$\hat{\rho} = \hat{x}_3 + \frac{x_2}{\hat{x}_1}. \tag{14}$$

Figure 9 shows the evolution of the synchronized response where the fast and slow convergences of both $\hat{x}_1$ and $\hat{x}_3$, respectively, are explained by the ratio σ/β. In contrast to the previous case, the steady state offset in $\hat{x}_2$ is persistent and cannot be made zero via tuning the feedback gain, k.

A similar argument applies for the case when x_3 is used as the driving signal for synchronization. Equation (15) illustrates the dynamics of the response system, where β_n is the nominal value for β,

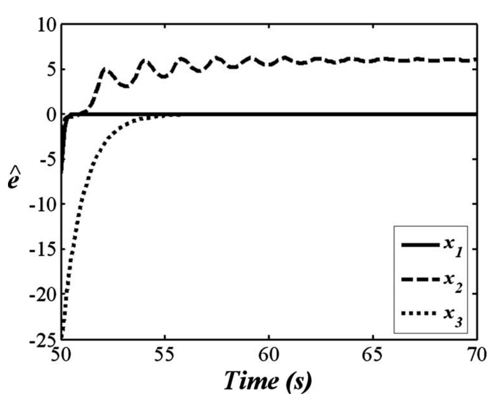

FIG. 9. Case study II: using x_2 for synchronization.

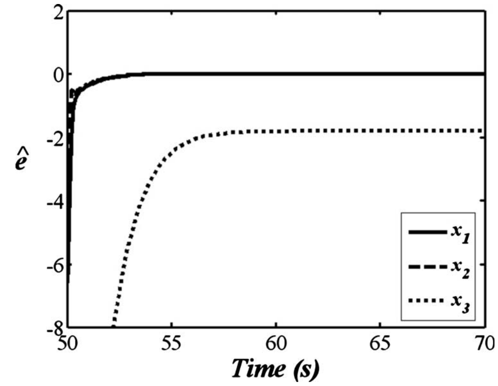

FIG. 10. Case study III: using x_3 for synchronization.

$$\dot{\hat{x}}_1 = -\sigma\hat{x}_1 + \sigma\hat{x}_2 + kx_3,$$

$$\dot{\hat{x}}_2 = \rho\hat{x}_1 - \hat{x}_2 - \hat{x}_1 x_3,$$

$$\dot{\hat{x}}_3 = -\beta_n\hat{x}_3 + \hat{x}_1\hat{x}_2. \tag{15}$$

Equation (16) and Fig. 10 illustrate the asymptotic and dynamic behavior of the synchronization errors, respectively,

$$\hat{x}_1 \to x_1, \quad \hat{x}_2 \to x_2,$$

$$e_{3ss} = (\hat{x}_2 - x_2)_{ss} = \left(\frac{1}{\beta_n} - \frac{1}{\beta}\right) x_{1ss} x_{2ss},$$

$$\hat{\beta} = \frac{\hat{x}_1\hat{x}_2}{x_3}. \tag{16}$$

V. EXTENSION TO OTHER CHAOTIC SYSTEMS

Complete and partial identification proved to be strongly dependent on the structure of the mathematical model of the system and the availability of the measured states. In addition, stabilizing the chaotic system to a stable steady state with and without synchronization-based state observers requires avoiding equilibrium points for which identifiability is lost. This suggests that the application of the proposed technique requires careful choice of the feedback structure and *a priori* knowledge of the behavior of the uncontrolled chaotic system. In the following, the case of partial identification of both the Rössler and Chua systems are investigated to examine the possibility of extending the proposed technique for other chaotic systems.

A. Partial identification of the Rössler equation

The Rössler system is yet another benchmark model when considering chaos. It was originally developed to model turbulence in fluids [33] and recent research showed how it can be controlled using both model- and nonmodel-based approaches [25]. It was also considered, along with other chaotic system, in applications dealing with synchronization of nonidentical system [32]. Equation (17) shows the dynamic model of the Rössler system, where it is assumed that only b is considered unknown. It is seen that the only nonlinear term appears in the equation of $\dot{x}_3$ and that, at equilibrium, there are only two possible steady state solutions.

$$\dot{x}_1 = -x_2 - x_3,$$

$$\dot{x}_2 = x_1 + ax_2,$$

$$\dot{x}_3 = -cx_3 + b + x_1x_3 + u, \tag{17}$$

where, again, $u=kx_i$, $i \in \{1,2,3\}$ and the nominal values for the parameters are $a=b=0.2$ and $c=5.7$. Examining the structure of the Rössler system shows that the feedback signal does not break the relationship between the states at steady state as $x_1=-ax_2$ and $x_3=-x_2$ regardless of the value of k. This fact greatly simplifies the partial identification process of the unknown parameter b as the knowledge of the available scalar time series of the observable signal is sufficient for arriving at $\hat{b}$. This highlights the fact that the identification process of the unknown and/or uncertain parameters is strongly dependent on the structure of the dynamical model at hand as, in contrast to Lorenz system, the Rössler system does not require state observers. At steady state, and assuming that the Rössler system will settle down to a stable identifiable equilibrium point, the estimate of b is given by

$$\hat{b} = \begin{cases} (x_1/a)(c - ak - x_1), & i = 1, \\ -x_2(c + k + ax_2), & i = 2, \\ x_3(c - k - ax_3), & i = 3, \end{cases} \tag{18}$$

where the value i corresponds to the only state being used for feedback. In addition, Eq. (18) suggests that the partial identification process could be easily extended to either a or c; thus, given the knowledge of any two parameters, the third one could be identified. Figure 11 shows the response of the identification process when the feedback controller $u=4x_1$ was switched on after 50 s. Using linear stability analysis at the equilibrium points, similar to the one carried out for the Lorenz system in Sec. III, it was found that the system could not be stabilized using the time series for either x_2 or x_3 for all values of k as all Lyapunov exponents included positive real parts.

B. Partial identification of the Chua equation

The last example to be considered is a Chua system with smooth cubic nonlinearity that is known to be a variant of the famous Chua system with piecewise linear characteristics [34] that is implementable in both analog and digital hardware and has typical applications in secure communications and synchronization-based applications [1,20]. The proposed structure of the Chua system along with both the feedback controller and synchronization-based state observer is given by Eq. (19), where it is assumed that only c is considered unknown.

$$\dot{x}_1 = \alpha(x_2 - x_1^3 + cx_1) + u,$$

$$\dot{x}_2 = x_1 - x_2 + x_3,$$

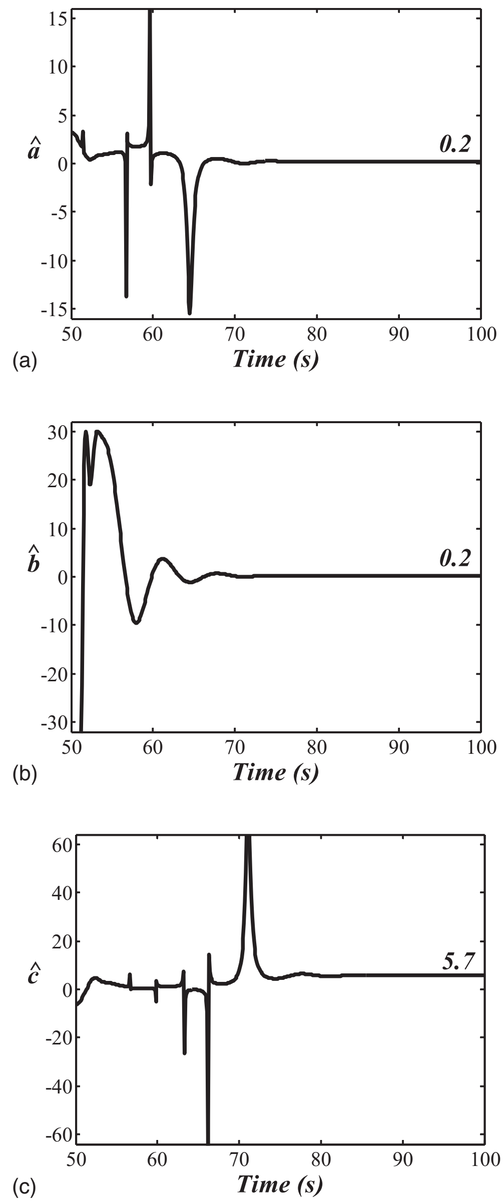

FIG. 11. Partial identification of the Rössler system.

$$\dot{x}_3 = -\beta x_2,$$

$$\dot{\hat{x}}_2 = x_1 - \hat{x}_2 + \hat{x}_3,$$

$$\dot{\hat{x}}_3 = -\beta\hat{x}_2, \tag{19}$$

where the nominal values for the parameters are $\alpha=10$, $\beta=16$, $c=0.2$, and it is assumed that only the time series for x_1 is available. Both x_2 and x_3 can be estimated using a synchronization-based state observer when x_1 is used as the drive signal for the response system constituting both $\hat{x}_2$ and $\hat{x}_3$. The stability of the response system in Eq. (19) can be verified using the Lyapunov function in Eq. (20) and the consequent result of Eq. (21).

$$L = \beta(\hat{x}_2 - x_2)^2 + (\hat{x}_3 - x_3)^2, \tag{20}$$

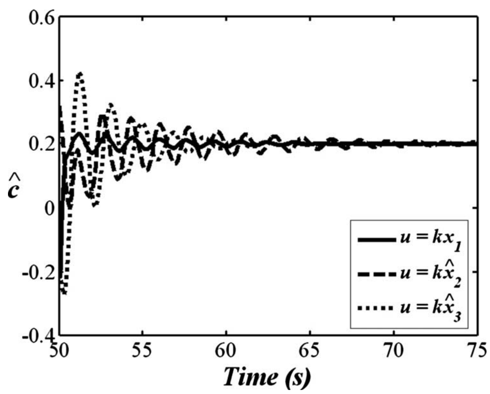

FIG. 12. Partial identification of $\hat{c}$ of the Chua system.

$$\dot{L} = 2\beta(\hat{x}_2 - x_2)(\dot{\hat{x}}_2 - \dot{x}_2) + 2(\hat{x}_3 - x_3)(\dot{\hat{x}}_3 - \dot{x}_3) = -2\beta(\hat{x}_2 - x_2)^2 < 0. \tag{21}$$

Using both steady-state analysis and linear stability techniques, it was found that the system can be successfully stabilized to an identifiable stable equilibrium point using either x_1, $\hat{x}_2$, or $\hat{x}_3$. The estimate of the unknown variable c, using $u=kx_i$, is given by

$$\hat{c} = \begin{cases} x_1^2 - k/\alpha, & i = 1, \\ x_1^2, & i = \hat{2}, \\ x_1^2 + k/\alpha, & i = \hat{3}. \end{cases} \tag{22}$$

Figure 12 illustrates the response of the partial identification process of c, where it is demonstrated that $\hat{c}$ did converge to its true value using $k=\{5,-5,-5\}$ for $\{x_1,\hat{x}_2,\hat{x}_3\}$, respectively.

VI. ADVANTAGES AND LIMITATIONS

The detailed analysis of the Lorenz system, carried out in Secs. II–IV illustrates that there are many possible implementations of the feedback controller of the form $u=\Sigma_i k_i x_i$ when using the full state feedback and $u=k_j x_j+\Sigma_{i\neq j} k_i \hat{x}_i$ when using the synchronization-based state observers. Only the case when a single state is used for feedback was considered to simplify the analysis and design of the proposed methodology. It should be noted, however, that some implementations can cause partial or complete loss of identifiability for some or all of the unknown parameters, e.g., when using either $u=kx_1$ in the equation of $\dot{x}_2$ or $u=kx_3$ in the equation of $\dot{x}_3$ the proposed method fails to identify σ. On the other hand, when considering the Rössler model, it was demonstrated that using state observers can be avoided when using the equation of $\dot{x}_3$ in implementing the feedback controller for any linear combination of the states, and it can be further proven that using a combination of both x_2 and x_3 in the equation of $\dot{x}_1$, or a combination of both x_1 and x_2 in the equation of $\dot{x}_2$ will have the same result. As for the Chua model, the feedback controller must be implemented using the equation of $\dot{x}_1$, otherwise α will be unidentifiable. Thus, which state is used for feedback and which parameter is to be identified, crucially affect the design process.

Using a single state to construct the feedback controller greatly simplifies the design and implementation of the proposed identification method. It was demonstrated that the tuning effort for a single gain is significantly less compared to other tuning methods currently reported in the literature for stabilizing chaotic systems to either periodic orbits or steady states, e.g., time delay autosynchronization and notch filters feedback [35,36]. In addition, using a single linear feedback maintains the same order of the system in the case of using the full state feedback, while the order increases by the same amount as the number of the observed states when using synchronization-based state observers [19,37,38]. The absence of parameter update laws for the identified parameters is shown to maintain simplicity of the design and ease of implementation either in analog or digital hardware. Since the estimates of the identified parameters converge to their true values only after the system reaches a steady state, the identification algorithm can be invoked after a certain delay time from switching on the feedback stabilizing controller to avoid any unwanted transient effects resulting from singularities and/or ill-conditioned steady states. This adds more flexibility and versatility to the proposed design and ensures practicality of the implementation.

To add more value to the discussion regarding advantages and limitations of the proposed methodology, the case of tracking slow changes in the unknown parameter(s), while

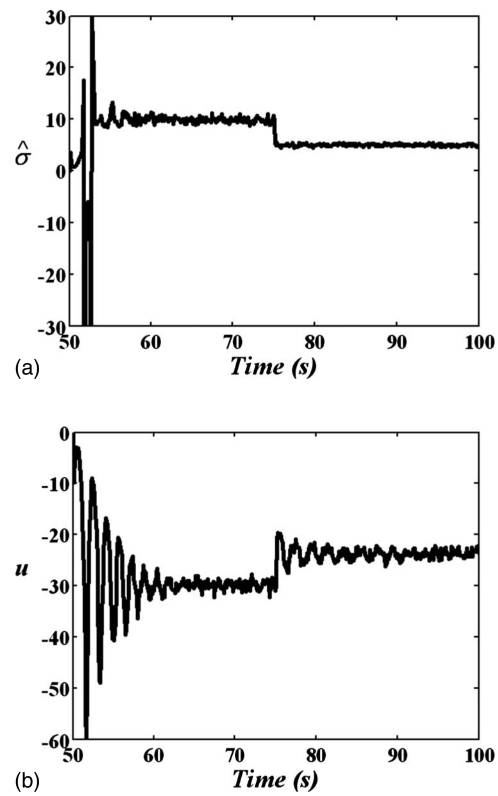

FIG. 13. Tracking while having noisy feedback for the Lorenz system.

having a noisy measurement of the state, used to construct the feedback controller, is now investigated. Considering the Lorenz system, and using the case where $u=-10x_1$ for $t \geqslant 50$ s in the equation of $\dot{x}_1$, Fig. 13 shows the response of the identification algorithm when σ was allowed to change from 10.0 to 5.0 at $t=75$ s while the feedback signal is subjected to a white noise of zero mean and 10% variance. As illustrated in Fig. 13(a) the proposed algorithm was able to quickly track the changes in the unknown parameter while maintaining satisfactory performance regardless of the fluctuations in the control signal, shown in Fig. 13(b). Thus the identification process is robust. Further investigations of both the Rössler and Chua systems was carried out and a similar result was obtained ensuring the superiority and versatility of the identification method, however, generalization of the result to other chaotic systems needs further investigation.

VII. SUMMARY AND CONCLUSION

A technique for both complete and partial identification of unknown and/or uncertain parameters of chaotic system was introduced. This technique does not require designing dynamic parameter update laws that are proven to be difficult and, in addition, are known to increase the order of the overall system. The proposed design was simple, yet efficient as demonstrated by the simulation results for three different chaotic systems, namely, Lorenz, Rössler, and Chua models. When using the full state vector, the design procedure reduces to two successive steps; first, based on the structure of the chaotic system at hand, a single state feedback is inserted to one of the dynamic equations of the model such that the steady-state solution involves the unknown and/or uncertain parameter(s) to be identified. Second, a simple linear stability analysis is carried out for the resulting steady state to decide on the range of the feedback gain that guarantees both stability and identifiability. The functionality of the identification process was extended to include state observers to account for practical situations where only a single scalar time series is available for feedback. In addition, it was shown that the identification process is robust in terms of the ability to track slow variations of the unknown parameter(s) in the presence of noisy feedback signals. The effect of identifying the uncertain parameters on the process of complete synchronization was also investigated, for the case of the Lorenz system, and it was shown that it is possible to resolve the conflict between partial identification and synchronization for the special cases when the uncertain parameter appears in the dynamic equation of the observable state. A generalization of the obtained results for other chaotic system is not possible because of the strong dependence of the proposed technique on the structure of the model at hand, e.g., complete identification of the parameters of the Rössler system was possible without having to construct state observers, while for the Chua model this was not possible. It is believed that the proposed technique can be applied for real-time applications, e.g., secure communications, via performing the identification (for the transmitter) and consequently the synchronization (between the receiver and transmitter and receiver) processes in sequence before masking the signal. The possibility to improve the performance of the suggested technique needs further exploration, e.g., replacing synchronization-based state observers with other techniques that allow controlling the rate of convergence of the observed states, or using nonlinear feedback to improve the transient performance of the stabilizing controller. Tackling these issues along with studying sensitivity to initial conditions and meeting constraints regarding the maximum control effort is under progress.

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